1. (Model for a glueball) Construct the color singlet combination of two gluons. One method is as follows: Let

\[ c = \begin{pmatrix} R \\ G \\ B \end{pmatrix}. \]

Under \( SU(3) \), \( c \rightarrow c' = Uc \), where \( U \) is a unitary (\( UU^\dagger = 1 \)) matrix of determinant 1. Similarly, let \( d^\dagger = (\bar{R}, \bar{G}, \bar{B}) \), transforming under the rule \( d^\dagger \rightarrow d'^\dagger = d^\dagger U^\dagger \). Form the matrix

\[ M \equiv cd^\dagger = \begin{pmatrix} R\bar{R} & R\bar{G} & R\bar{B} \\ G\bar{R} & G\bar{G} & G\bar{B} \\ B\bar{R} & B\bar{G} & B\bar{B} \end{pmatrix}. \]

Note that \( M' = c'd'^\dagger = UMU^\dagger \). Remove the trace:

\[ G \equiv M - \frac{1}{3} [Tr(M)], \]

so that \( Tr(G) = 0 \). The question is how to put together two octets to make a singlet; that is, how to make something bilinear in \( G_1 \) and \( G_2 \) which is invariant under \( U \). The solution is \( s \equiv Tr(G_1G_2) \). This is kind of the invariant product of two octets, i.e., the \( SU(3) \) analog to the dot product of two 3-vectors in \( SU(2) \).

2. (Subprocess for gluon emission, H&M 10.4) Derive

\[ \frac{d\sigma}{dp_T^2} \approx \frac{1}{16\pi \hat{s}^2} |\mathcal{M}|^2. \]

Make use of H&M (10.26) and (10.20), together with (4.34). Show that the \( \gamma^*q \) flux factor is given by \( 2\hat{s} \) using the convention of (8.48).

3. (Gluon-gluon scattering)
   (a) Draw the lowest-order diagrams (there are four of them) representing the interaction of two gluons.
   (b) Write down the corresponding amplitudes.
   (c) Put the incoming gluons in the color singlet state; do the same for the outgoing gluons. Compute the resulting amplitudes.
   (d) Go to the \( CM \) frame, in which each gluon has energy \( E \); express all the kinematic factors in terms of \( E \) and the scattering angle \( \theta \). Add the amplitudes to get the total, \( \mathcal{M} \).