BIEMBEDDINGS OF $K_{13}$

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Abstract. Using exhaustive searches all decompositions of $K_{13}$ into two graphs which can be embedded on the torus or on the Klein bottle are obtained.

1. Introduction

A graph which can be embedded on a surface with Euler characteristic $\varepsilon$ and which has $n$ vertices has at most $3(n-\varepsilon)$ edges. So a graph which can be embedded on the torus or on the Klein bottle and which has 13 vertices has at most 39 edges. Suppose that the graph $K_{13}$ which has 78 edges is the union of two graphs each of which can be embedded on a surface with Euler characteristic 0. Since this union would provide just enough edges for $K_{13}$ each graph must triangulate the surface on which it can be embedded and each graph is the complement of the other.

Ringel [4], Beineke [1], and Jackson and Ringel [3] have provided various constructions which show that $K_{6n+1}$ is the union of $n$ copies of a graph which can be embedded on the torus. In particular for $n=2$, these constructions provide a self-complementary graphs with 13 vertices which can be embedded on the torus. Borodin and Mayer [2] each constructed a graph with 13 vertices which can be embedded on the torus and whose complement can be embedded on the Klein bottle.

Jackson and Ringel [3] have conjectured that there does not exist a graph with 13 vertices which can be embedded on the Klein bottle and whose complement can also be embedded on the Klein bottle.

2. Searching

Exhaustive computer searches were performed to find all graphs with 13 vertices for which both the graph and its complement can be embedded on surfaces with Euler characteristic 0. As pointed out above any such graph and its complement must be triangulations of surfaces with Euler characteristic 0.

The steps of one of these searches is outlined here. Suppose we want to find all the graphs with 13 vertices which can be embedded on the Klein bottle and whose complement can be embedded on the torus. (The computer times given are on a 2+ GHz processor with 1 Gbytes of memory.)

- Generate all the graphs with 13 vertices which triangulate the Klein bottle. Write each graph in canonical form to a list $L_K$. The triangulations are generated by a program which starts with irreducible triangulations and splits vertices until the desired number of vertices is reached. This list has

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21,535,942 entries of which 20,741,047 are unique. (Generate triangulations: 94 sec; put graphs in canonical form: 142 sec)

- Sort $L_K$. (130 sec)
- Create a second list $L_T$ of the complements of all the graphs with 13 vertices which triangulate the torus. Write each complement in canonical form to $L_T$. This list has 8,778,329 entries of which 8,447,983 are unique. (Generate triangulations: 38 sec; find complements: 28 sec; put complements in canonical form: 56 sec)
- Sort $L_T$. (44 sec)
- Compare the two lists $L_K$ and $L_T$ for common entries. (11 sec)

3. Figures

The figures of triangulations shown here consist of four parts. On the left of each figure is a polygon representing the triangulation. The rotation is given in the upper right of each figure. Below the rotation is a list of generators for the automorphism group of the triangulation. If the automorphism group is trivial then no generators are shown. In the lower right of each figure is the degree sequence of the embedded graph which is a list of the degrees of the vertices in non-decreasing order.

4. Torus - Torus Biembeddings

An exhaustive search found 22 graphs with 13 vertices which can be embedded on the torus and whose complements can also be embedded on the torus. Of these 22 graphs 6 are self-complementary and the other 16 graphs form 8 pairs of non-isomorphic complementary graphs. One of these graphs can be embedded on the torus in two different ways. Triangulations on the torus using these 22 graphs are shown in Figs. 5, 7, 9, 11, 13, 15, 17, 18, 19, 20, and 23 through 34. That these 22 graphs are unique can be seen from their degree sequences except for Figs. 33 and 34. But these two are not isomorphic because in Figure 33 vertex $f$, the only vertex with degree 4, has four neighbors of degree 9 while in Figure 34 vertex $h$, the only vertex in this graph with degree 4, has two neighbors of degree 9.

Figures 5 through 34 show the 15 ways the edges of $K_{13}$ can be decomposed into two subgraphs both of which can be embedded on the torus.

These decompositions are denoted as tt1 through tt15. Decompositions tt1 through tt6 consist of six self-complementary graphs and their isomorphic complements. Decomposition tt1 is equivalent to the one generated by the method described by Ringel in [4]. Decomposition tt2 is equivalent to the one generated by Beineke in [1]. Jackson and Ringel [3] also produce this decomposition using the current graph reproduced here in Figure 1.

If we modify in turn the direction of rotation of each of the vertices in Figure 1 we get the current graphs in Figs. 2, 3, and 4. These current graphs produce tt3, tt4, and tt1.

The pairs of graphs used in tt8 and tt9 are the same. The triangulations in Figs. 19 and 21 are the same. The graphs in Figs. 20 and 22 are the same but they are embedded differently. To see this consider the two faces $akd$ and $bcl$ in each triangulation which are the only faces with vertices of degree $\{3,9,9\}$. In Figure 20 both of these faces are adjacent to faces with vertices of degree $\{3,6,9\}$, $\{3,6,9\}$, and $\{6,9,9\}$ while in Figure 22 these faces are adjacent to faces with vertices of


5. Torus - Klein bottle biembeddings

An exhaustive search found 18 graphs with 13 vertices which can be embedded on the torus and whose complements can be embedded on the Klein bottle. Triangulations of these graphs on the torus and triangulations of their respective complements on the Klein bottle are shown in Figs. 35 through 70.

All 18 graphs are unique. This can be seen from the degree sequences with the exceptions of two pairs of graphs. Graphs $tk4t$ and $tk7t$ in Figs. 41 and 47 have the same degree sequence, however, the one vertex of degree 5, $l$, in Figure 41 is adjacent to a vertex of degree 3 while the one vertex of degree 5, $k$, in Figure 47 is not adjacent to a vertex of degree 3. Graphs $tk11t$ and $tk16t$ in Figs. 55 and 65 also have the same degree sequence. Consider the set of vertices in each of these two graphs which are adjacent to two vertices of degree 3. In $tk11t$ these vertices are $\{j, k, l\}$ while in $tk16t$ they are $\{j, l, m\}$. If these two graphs were isomorphic then the isomorphism must map $\{j, k, l\}$ of $tk11t$ onto $\{j, l, m\}$ of $tk16t$. But in
tk11t b which has degree 3 is adjacent to each element of \{j, k, l\} while in tk16t each vertex of degree 3 is adjacent to only two elements of \{j, l, m\}.

6. **Klein bottle - Klein bottle biembeddings**

An exhaustive search found no graphs with 13 vertices which can be embedded on the Klein bottle and whose complement can also be embedded on the Klein bottle. This confirms the conjecture of Jackson and Ringel [3].

**References**

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Figure 5. \texttt{tt1a}

Figure 6. \texttt{tt1b}
Figure 7. $tt2a$

Figure 8. $tt2b$
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Figure 9. tt3a

Figure 10. tt3b
Figure 11. tt4a

Figure 12. tt4b
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Figure 13. tt5a

Figure 14. tt5b
Figure 15. tt6a

Figure 16. tt6b
Figure 17. tt7a

Figure 18. tt7b
Figure 19. tt8a

Figure 20. tt8b
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Figure 21. tt9a

(a)(b)(c)(d)(e)(f)(g)(h)(i)(j)(k)

Figure 22. tt9b

(a)(b)(c)(d)(e)(f)(g)(h)(i)(j)(k)
Figure 23. tt10a

Figure 24. tt10b
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Figure 25. tt11a

Figure 26. tt11b
Figure 27. tt12a

Figure 28. tt12b
Figure 31. tt14a

Figure 32. tt14b
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Figure 33. tt15a

Figure 34. tt15b
Figure 35. tk1t Torus of O.V. Bordin

Figure 36. tk1k Klein bottle of O.V. Bordin
Figure 37. $tk2t$ Torus of J. Mayer

Figure 38. $tk2k$ Klein bottle of J. Mayer
Figure 39. tk3t

Figure 40. tk3k
Figure 41. tk4t

Figure 42. tk4k
Figure 43. tk5t

Figure 44. tk5k
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Figure 45. tk6t

Figure 46. tk6k
Figure 47. tk7t

Figure 48. tk7k
BIEMBEDDINGS OF $K_{13}$

Figure 49. tk8t

Figure 50. tk8k
Figure 51. tk9t

Figure 52. tk9k
BIEMBEDDINGS OF $K_{13}$

Figure 53. tk10t

Figure 54. tk10k
Figure 55. tk11t

Figure 56. tk11k
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Figure 57. tk12t

Figure 58. tk12k
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Figure 61. tk14t

Figure 62. tk14k
Figure 63. tk15t

Figure 64. tk15k
Figure 67. tk17t

Figure 68. tk17k
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Figure 69. tk18t

Figure 70. tk18k

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